Box Progressions and Abelian Power Free Words

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The Setup

Given balls and boxes both enumerated with positive integers, we consider a sequential allocation of the balls into the boxes.

We fix an $\ell \geq 2$. Proceeding in increasing order of box labels, choose an arbitrary integer r between 1 and ℓ, and assign to each box the next r smallest unassigned balls.

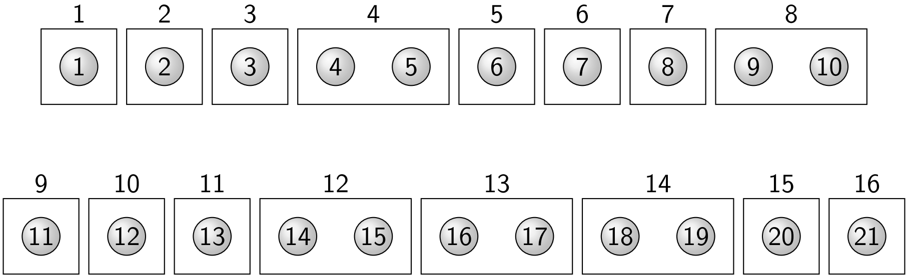
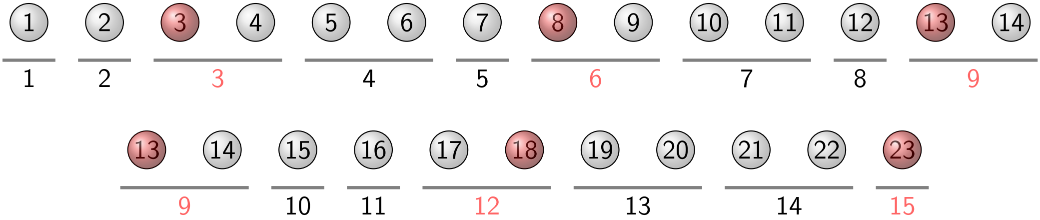


Figure 1: An example of a 2-ball-box distribution

- We focus on ball and box labels that simultaneously form a k-term arithmetic progression.
- We call these balls and boxes k ball box progressions (k-BPs).



Balls: $3, 8, 13, 18, 23$ Boxes: $3, 6, 9, 12, 15$

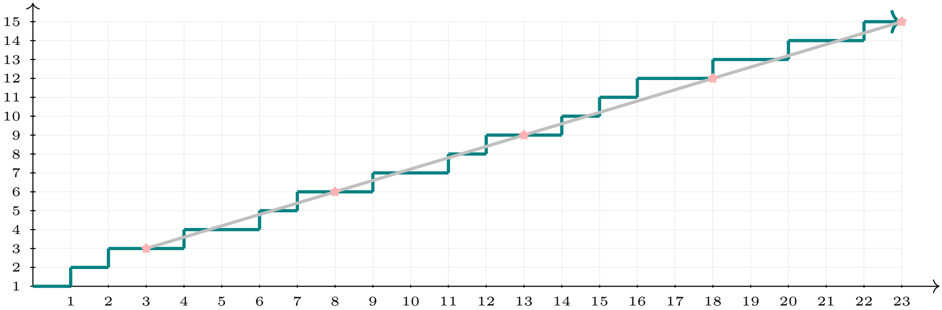
Question

- We investigate the minimal number of balls needed to guarantee the existence of ball-box progressions
- Is there such a minimal number for every k?
- Or are there infinitely long ball-box distributions with no k ball-box progressions?

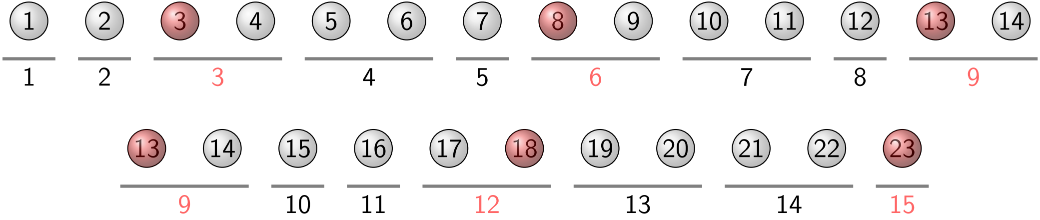
- $l = 2$. First non-trivial case
- The problem seems related to Ramsey theory at first.

Plan Of The Talk

Equivalent Formalization- I



$(3,3)\,,\,(8,6)\,,\,(13,9)\,,\,(18,12)\,,\,(23,15)$



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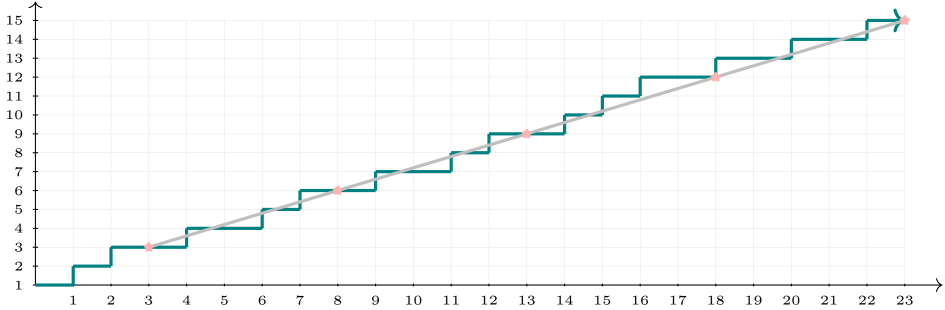


$$(3,3)\,,\,(8,6)\,,\,(13,9)\,,\,(18,12)\,,\,(23,15)$$

- Indeed, we are looking for arithmetic progressions in $\mathbb{N} \times \mathbb{N}$ in the graph above.
- The analogy with Szemerédi is suggestive

Theorem (Multidimensional Szemerédi Theorem on $\mathbb{N}^2$)

Let $A \subset \mathbb{N}^2$ be a subset with positive upper density:

$$\overline{d}(A) = \limsup_{N \to \infty} \frac{|A \cap [1, N]^2|}{N^2} > 0.$$

Then, for every $k \in \mathbb{N}$, there exist $x \in \mathbb{N}^2$ and nonzero $v \in \mathbb{N}^2$ such that

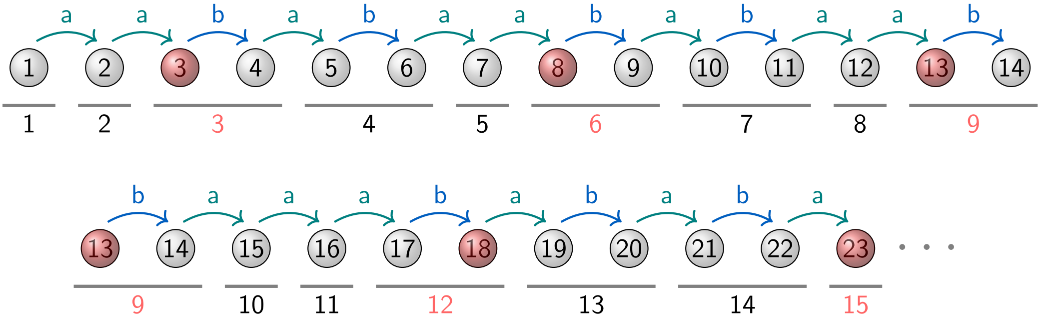
$$\{x, x+v, x+2v, \ldots, x+(k-1)v\} \subset A.$$

- The density of integer points on the graph is $O\left(\frac{1}{N}\right)$. This is not a dense subset.
- Multidimensional Szemerédi is not applicable, another formalization?

Equivalent Formalization- II

We can encode this sequence into a word over $\Sigma := \{a, b\}$ via the following rules:

- an a if the box changes
- a b if the same box



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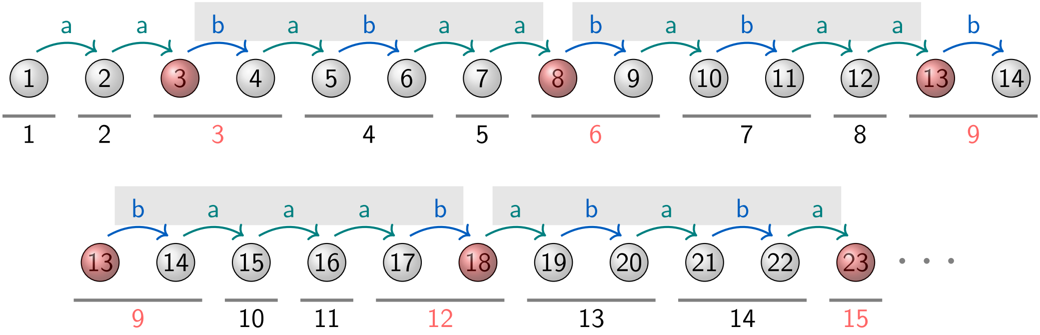


Combinatorics on Words

Repetition

A k repetition is composed of k consecutive equal factors. Examples:

- $aaaa$
- $\underline{aaabb}\ \underline{aaabb}\ \underline{aaabb}\ \underline{aaabb}$

Abelian Power

A k abelian power is composed of k consecutive factors that are anagrams of each other. Examples:

- $aaaa$
- $\underline{aaabb}\ \underline{aaabb}\ \underline{aaabb}\ \underline{aaabb}$
- $angle$, $angel$
- $\underline{babaa}\ \underline{babaa}\ \underline{baaab}\ \underline{ababa}$

Ball Box Progressions and Abelian Powers

Proposition

Let $k \geq 2$ be a positive integer. Let $\mathcal{D}$ a ball-box distribution, the corresponding word $w_{\mathcal{D}}$ is abelian $(k-1)$-power free if and only if there is no k ball-box progression in $\mathcal{D}$.

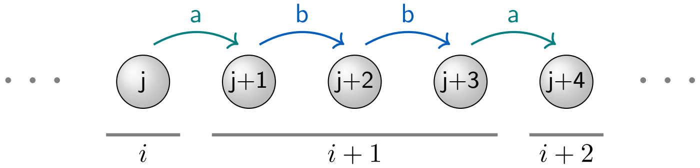
being k ball-box progression free $\longleftrightarrow$ being $k-1$ abelian power free

Some previous work on abelian power-free words:

- F. Dekking, Strongly non-repetitive sequences and progression-free sets, Journal of Combinatorial Theory 27 (1979)
- A. Carpi, On abelian power-free morphisms, Internat. J. Algebra Comput. 3 (1993) 151–167.

Our Constraint On The Word

at most 2 balls in a box $\longleftrightarrow$ no bb as a factor



Some Notions on Words-I

- Let $\Sigma = \{a, b\}$ be an alphabet. An element of Σ^* is called a word.
- The empty word is denoted by ϵ.
- Given a word
$$v = uw,$$
u is called a prefix.
- We set
$$\mathrm{Pref}(v) = \{u \in \Sigma^* : u \text{ is a prefix of } v\}.$$
- The Parikh vector of v denoted by
$$\psi(v) = (v_1, v_2) \in \mathbb{N}^{|\Sigma|}$$
where v_1 and v_2 is the number of appearances of a and b respectively.

Some Notions on Words-II

- A morphism is a map

$$\begin{aligned} h: \quad \Sigma^* &\to \Sigma^* \\ h(vw) &\mapsto h(v)h(w). \end{aligned}$$

- The frequency matrix of a morphism is

$$\begin{pmatrix} \psi(h(a)) \\ \psi(h(b)) \end{pmatrix}.$$

- Finally, we define

$$\mathrm{Pref}\,(h) := \bigcup_{\sigma \in \Sigma} \mathrm{Pref}(h(\sigma)).$$

- h is called abelian k-power-free if, for every abelian k-power free word $w \in \Sigma^*$, the image $h(w)$ is also ableian k-power free.

Example

- Consider the morphism $h : \Sigma^* \to \Sigma^*$ given by

$$\begin{aligned} h(a) &= aaaba \\ h(b) &= bab. \end{aligned}$$

- The frequency matrix of h is

$$M = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

  The determinant of the matrix is 7.

-
$$\text{Pref}(h) = \{\epsilon, a, aa, aaa, aaab, b, ba\}$$

Question

- Is h a k abelian power free morphism for some k?

Some Observation on h

- Define the group generated by $\psi(h(a)) = (4,1)$ and $\psi(h(b)) = (1,2)$.

$$G_h = \langle \psi(h(\sigma)) : \sigma \in \Sigma \rangle \subseteq \mathbb{Z}^\Sigma.$$

$$\mathrm{Pref}(h) = \{\epsilon, a, aa, aaa, aaab, b, ba\}$$

$$\mathcal{P} := \psi(\mathrm{Pref}(h)) = \{(0,0), (1,0), (2,0), (3,0), (3,1), (0,1), (1,1)\}.$$

- We observe that $(0,1)$ and $(1,2)$ are in the same coset in $\mathbb{Z}^\Sigma / G_h$:

$$(3,0) - (0,1) = (4,1) - (1,2).$$

$$\overline{\mathcal{P}} = \{\overline{(0,0)}, \overline{(1,0)}, \overline{(2,0)}, \overline{(3,0)}, \overline{(1,1)}, \overline{(3,1)}\}.$$

Arithmetic Rank (a-rk)

The arithmetic rank of a set A is defined as

$$\text{a-rk}(A) := \text{length of the longest arithmetic progression in } A$$

- Example: $\text{a-rk}(\psi(\mathrm{Pref}(h))) = 6$

Theorem (Eyidoğan, Göral, T.)

Given an integer $n \geq 4$, an alphabet Σ of size at least 2 and a morphism

$$h : \Sigma^* \to \Sigma^*.$$

Let

$$G_h = \langle \psi(h(\sigma)) : \sigma \in \Sigma \rangle \subseteq \mathbb{Z}^\Sigma.$$

Finally we set

$$\mathcal{P} = \psi(\mathrm{Pref}\,(h)) \text{ and } \overline{\mathcal{P}} = \{\overline{v} : v \in \mathcal{P}\}.$$

Suppose the following conditions are satisfied:

1. The frequency matrix of h is non-singular,
2. $\text{a-rk}(\overline{\mathcal{P}}) \leq n$,
3. For $v \in \mathcal{P} \setminus (\psi(h(\Sigma)) \cup \{0\})$, the equivalence class of v contains at most 2 elements. Moreover if v and w are in the same class, then

$$v - w = \psi(h(p)) - \psi(h(q))$$

   for some $p, q \in \Sigma$ where $v \in \psi(\mathrm{Pref}(h(p)))$ and $w \in \psi(\mathrm{Pref}(h(q)))$.

Then, h is an abelian n-power free morphism.

Corollary

There exists an infinite word Ω over a binary alphabet $\Sigma = \{a, b\}$ such that

- bb is not a factor of Ω (there are no two consecutive b's in Ω),
- Ω is abelian 6-power free.

- We consider the morphism

$$\begin{aligned} h(a) &= aaaba \\ h(b) &= bab. \end{aligned} \quad \text{with non-singular frequency matrix} \quad \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}.$$

- $\text{a-rk}(\overline{\mathcal{P}}) = 6$
- We have only two elements in the same equivalence class:

$$(3,0) - (0,1) = (4,1) - (1,2).$$

and

$$(1,2) - (0,1) = (4,1) - (3,0) = \psi(h(a)) - \psi(h(b)).$$

- h is an abelian power-free morphism, hence $\Omega := h^{\omega}(a)$ is an infinite abelian 6 power-free word.

Answer

- Ω admits no 6 abelian powers
- Among the ball-box distributions where we allow at most 2 balls in a box, there is an infinitely long ball-box distribution corresponding to Ω which admits no 7 ball box progressions.